# Rainbow spanning subgraphs in bounded edge–colourings of graphs with large minimum degree

Cano, Pilar; Perarnau, Guillem; Serra, Oriol

*DOI:*
10.1016/j.endm.2017.06.039



*Citation for published version (Harvard):*
Cano, P, Perarnau, G & Serra, O 2017, 'Rainbow spanning subgraphs in bounded edge–colourings of graphs with large minimum degree', *Electronic Notes in Discrete Mathematics*, vol. 61, pp. 199-205.
https://doi.org/10.1016/j.endm.2017.06.039

# Rainbow spanning subgraphs in bounded edge–colorings of graphs with large minimum degree

Pilar Cano* [1,2], Guillem Perarnau† [3], and Oriol Serra‡ [1]

[1] *Departament de Matemàtiques Universitat Politècnica de Catalunya, Barcelona, Spain.*
[2] *School of Computer Science, Carleton University, Ottawa, Canada.*
[3] *School of Mathematics, University of Birmingham, Birmingham, United Kingdom.*

**Abstract**

We study the existence of rainbow perfect matching and rainbow Hamiltonian cycles in edge–colored graphs where every color appears a bounded number of times. We derive asymptotically tight bounds on the minimum degree of the host graph for the existence of such rainbow spanning structures.

## 1 Introduction

A subgraph $H$ of an edge-colored graph $G$ is *rainbow* if the edges of $H$ have pairwise distinct colors. Erdős and Spencer [6] initiated the study of rainbow spanning structures motivated by a problem of Ryser on the existence of Latin transversals in Latin squares [11]. By introducing the so–called Lopsided Lóvasz local lemma, they showed that every edge–coloring of the complete bipartite graph $K_{n,n}$ admits a rainbow perfect matching provided that each color appears at most $k \le (n-1)/(4e)$ times. Motivated by this result, an edge–coloring of $G$ is said to be *k–bounded* if every color appears at most $k$ times on the edges of $G$.

Albert, Frieze and Reed [1] proved the existence of rainbow Hamiltonian cycles in $k$-bounded edge-colored complete graphs $K_n$, provided that $k \le (n-1)/64$ (see also [7] for a clarification about the constant in the upper bound).

In a more general setting, Böttcher, Kohayakawa and Proccaci [3] proved that for every graph $H$ on $n$ vertices and maximum degree at most $\Delta$, there exists a rainbow copy of $H$ in any $k$–bounded coloring of the complete graph $K_n$, provided that $k \le n/(51\Delta^2)$. Their approach used the framework developed by Lu and Székely [10] on negatively dependency graphs constructed from matchings.

Other related results include the existence of almost spanning rainbow trees in $K_n$ [7], the existence of spanning rainbow multipartite graphs with bounded degrees in multipartite complete graphs $K_{n,n\dots,n}$ [8] as well as the existence of rainbow subhypergraphs in complete hypergraphs [5, 8]. All these results hold provided that the edge–coloring of the host graph is $k$-bounded for an appropriate value of $k$ and the proofs rely on variations of the Lóvasz local lemma.

*Email: m.pilar.cano@upc.edu.
†Email: g.perarnau@bham.ac.uk.
‡Email: oriol.serra@upc.edu.

A natural question is whether one can still find a rainbow copy of a spanning subgraph $H$ in a host graph $G$ that is sparser than the corresponding complete graph structure (complete graphs, complete multipartite graphs, complete hypergraphs ...). Obviously, a necessary condition is that $G$ must contain at least one copy of $H$. Among other conditions, the existence of a spanning subgraph can be guaranteed by requiring $G$ to have a large minimum degree.

Sudakov and Volec [13] gave a first step in this direction by proving the following (see also Kamčev, Sudakov and Volec [8, Theorem 1.7] for a formal statement).

**Theorem 1** ([13])**.** *There exist positive constants $\beta$ and $c$ such that the following holds. Let $H$ be a graph on $n$ vertices of maximum degree $\Delta$ and $G$ a graph on $n$ vertices and minimum degree $\left(1-\frac{c}{\Delta}\right)n$. Then any $k$-bounded edge–coloring of $G$ has a rainbow copy of $H$, provided that $k \leq \frac{n}{\beta\Delta^2}$.*

In this paper we study the existence of rainbow copies of $H$ in $G$ with asymptotically tight conditions on the minimum degree of $G$, in two particular cases: (1) $H$ is a perfect matching and $G$ is balanced and bipartite, and (2) $H$ is a Hamiltonian cycle and $G$ is arbitrary. In both cases the minimum degree threshold is $n/2$, which is also the minimum degree for the existence of a copy of the considered spanning subgraphs as follows from classical results.

Our first result states that, provided there are not too many edges with the same color, at the moment the minimum degree ensures the existence of a perfect matching in a bipartite graph, it also ensures the existence of a rainbow one.

**Theorem 2.** *For every $\epsilon > 0$, there exist $c > 0$ and $n_0$ such that for every $n \geq n_0$ the following holds: If $G = (A, B)$ is a bipartite graph with $|A| = |B| = n$ and minimum degree at least $(1+\epsilon)\frac{n}{2}$, then every $cn$-bounded edge–coloring of $G$ contains a rainbow perfect matching.*

Theorem 2 has a direct consequence on Latin squares. A *Latin square* $L$ of size $n$ is an $n \times n$ array with positive integer entries such that each number appears at most once in each row or column. A *partial* Latin square is a latin square where some positions are blank. A *transversal* of $L$ is a set of $n$ cells in the array, no two in the same row or column. A transversal is *Latin* if the integers contained in its non-blank cells are pairwise distinct.

**Corollary 3.** *For every $\epsilon > 0$, there exist $c > 0$ and $n_0$ such that for every $n \geq n_0$ the following holds: If $L$ is a partial Latin square of size $n$ where each row and column have at least $(1+\epsilon)\frac{n}{2}$ entries and no entry appears more than $cn$ times, then $L$ contains a Latin transversal.*

Our second result deals with the existence of rainbow Hamiltonian cycles in sparse graphs. Again the minimum degree required for the existence of a Hamiltonian cycle ensures the existence of a rainbow one, provided that each color does not appear too many times.

**Theorem 4.** *For every $\epsilon > 0$, there exist $c' > 0$ and $n_0$ such that for every $n \geq n_0$ the following holds: If $G$ is a graph on $n$ vertices with minimum degree at least $(1+\epsilon)\frac{n}{2}$, then every $c'n$-bounded edge-coloring of $G$ contains a rainbow Hamiltonian cycle.*

The main ingredient of our proofs is the study of the uniform measure on the set of perfect matchings (or of Hamiltonian cycles) of $G$. In contrast to the results for complete graphs, here the lack of symmetry poses a challenge to prove properties of typical spanning subgraphs. By performing a counting argument directly on $G$, we obtain an asymptotically tight result.

We make the following remarks on our results.

1) If $n$ is even, then Theorem 4 also provides the existence of a rainbow perfect matching in the graph. This does not supersede the result of Theorem 2 as the minimum degree condition there is much weaker.

2) The condition on the minimum degree in both theorems is asymptotically tight, since there are bipartite graphs with minimum degree $\lceil n/2 \rceil - 1$ and no perfect matching, as well as graphs with minimum degree $\lceil n/2 \rceil - 1$ and no Hamiltonian cycle. However, the dependence on the number of colors is not best possible. The use of the local lemma forces us to pay a constant factor. The dependence of $c$ in $\epsilon$ is $c = O(\epsilon^2)$ for Theorem 2 and $c = O(\epsilon^3)$ for Theorem 4. As a corollary, we obtain that Theorem 2 and 4 are still valid for minimum degree at least $\frac{n}{2} + O(\sqrt{n})$ and $\frac{n}{2} + O(n^{2/3})$, respectively, provided that the coloring is 2-bounded. We conjecture the exact version of our results:

   **Conjecture 5.** *There exists $c > 0$, such that the following holds:*

   - *any $cn$-bounded coloring of a balanced bipartite graph on $2n$ vertices and minimum degree at least $n/2$, admits a rainbow perfect matching.*
   - *any $cn$-bounded coloring of a graph on $n$ vertices and minimum degree at least $n/2$, admits a rainbow Hamiltonian cycle.*

3) As a corollary of Theorem 2, one can obtain the existence of linearly many disjoint rainbow perfect matchings on balanced bipartite graphs under the same minimum degree condition. A similar statement also holds for Hamiltonian cycles.

4) Theorem 1 in [13] sets a starting point for the study of rainbow subgraphs in graphs with large minimum degree. Define $\delta(\Delta, n)$ to be the infimum over all $\delta$ such that every graph $G$ on $n$ vertices and minimum degree at least $\delta n$ contains any graph $H$ on $n$ vertices and maximum degree $\Delta$. Let $\delta(\Delta) = \limsup_{n\to\infty} \delta(\Delta, n)$. Bollobás and Eldridge [2] and Catlin [4] independently conjectured that $\delta(\Delta) = 1 - \frac{1}{\Delta+1}$. The lower bound $\delta(\Delta) \geq 1 - \frac{1}{\Delta+1}$ is given by considering $H$ to be the union of $n/(\Delta+1)$ cliques and $G$ a slightly unbalanced $(\Delta+1)$-partite complete graph. While partial results have been obtained [9, 12], the conjecture is still wide open. Motivated by Theorems 2 and 4, we conjecture the sparse analogue of [3, Theorem 7]:

   **Conjecture 6.** *For every $\epsilon$ and $\Delta$, there exist $c > 0$ and $n_0$ such that for every $n \geq n_0$ the following holds: If $G$ is a graph on $n$ vertices with minimum degree at least $(1+\epsilon)\delta(\Delta)$ and $H$ is a graph on $n$ vertices with maximum degree at most $\Delta$, then every $cn$-bounded edge-coloring of $G$ contains a rainbow copy of $H$.*

# 2 Sketch of the proofs

We next give a sketch of the proof of our results. Let $\Omega$ be the space of perfect matchings of $G$ equipped with the uniform measure. Since $\delta(G) \geq n/2$, Hall's theorem ensures $\Omega \neq \emptyset$. Our goal is to apply the Lopsided version of the local lemma to a set of bad events $\mathcal{E}$ defined on $\Omega$. While these events do not exhibit bounded dependencies, the effect of events whose support does not intersect is weak and can be bounded.

We use the Lopsided version of the Lovász local lemma, which allows us to lower from above the probability of avoiding all bad events simultaneously, in terms of conditional probabilities.

**Lemma 7** (Lopsided Lovász Local Lemma [6]). *Consider a set $\mathcal{E}$ of (typically bad) events in a probability space $\Omega$. Let $P \in (0,1)$. For every $E \in \mathcal{E}$, consider a set $\mathcal{D}(E)$ of at most $D$ other events in $\mathcal{E}$. Suppose that for all $\mathcal{T} \subseteq \mathcal{E} \setminus (\{E\} \cup \mathcal{D}(E))$ we have that*

$$\mathbb{P}\left[E \mid \bigcap_{F \in \mathcal{T}} F^c\right] \leq P \ . \tag{1}$$

*If $4PD \leq 1$, then $\mathbb{P}\left[\bigcap_{E \in \mathcal{E}} E^c\right] > 0$.*

The main part of the proof consists on bounding $\mathbb{P}\left[E \mid \bigcap_{F \in \mathcal{T}} F^c\right]$. In order to do it, we use the switching method, a classic idea to bound probabilities in uniform combinatorial spaces. Let $\mathcal{P} \subseteq \Omega$ be a property of perfect matchings. We define a local operation that transforms perfect matchings that satisfy $\mathcal{P}$ into other perfect matchings in $\Omega \setminus \mathcal{P}$ and vice versa. By counting the number of operations that can be perform to each perfect matching, we get estimates on $\mathbb{P}[\mathcal{P}]$. In constrast to [6], the non-existence of edges has a strong effect on the number of defined operations.

The proof of Theorem 4 uses a similar approach with some variations. In this case, the natural switching operation can transform Hamiltonian cycles into other spanning 2-regular subgraphs. We use a density argument to justify that, in average, a positive proportion of operations produces Hamiltonian cycles. This suffices to bound (1).

## References


[1] M. Albert, A. Frieze, and B. Reed, *Multicoloured hamilton cycles*, Electronic Journal of Combinatorics **2** (1995), no. R10, 1–13.

[2] B. Bollobás and S. E. Eldridge, *Packings of graphs and applications to computational complexity*, Journal of Combinatorial Theory, Series B **25** (1978), no. 2, 105–124.

[3] J. Böttcher, Y. Kohayakawa, and A. Procacci, *Properly coloured copies and rainbow copies of large graphs with small maximum degree*, Random Structures & Algorithms **40** (2012), no. 4, 425–436.

[4] P. A. Catlin, *Embedding subgraphs and coloring graphs under extremal degree conditions*, Proc. 8th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, pp. 139–145.

[5] A. Dudek, A. Frieze, and A. Rucinski, *Rainbow Hamilton cycles in uniform hypergraphs*, Electronic Journal of Combinatorics **19** (2011), no. 1, P46.

[6] P. Erdős and J. Spencer, *Lopsided lovász local lemma and latin transversals*, Discrete Applied Mathematics **30** (1991), no. 2, 151–154.

[7] A. Frieze and M. Krivelevich, *On rainbow trees and cycles*, Electron. J. Combin **15** (2008), no. 1, 9.

[8] N. Kamčev, B. Sudakov, and J. Volec, *Bounded colorings of multipartite graphs and hypergraphs*, arXiv preprint arXiv:1601.02271 (2016).

[9] H. Kaul and A. Kostochka, *Extremal graphs for a graph packing theorem of Sauer and Spencer*, Combinatorics, Probability and Computing **16** (2007), no. 03, 409–416.

[10] L. Lu and L. Székely, *Using Lovász Local Lemma in the space of random injections*, Electron. J. Combin **14** (2007), no. 1, 13.

[11] H.J. Ryser, *Neuere probleme der kombinatorik*, (1967).

[12] N. Sauer and J. Spencer, *Edge disjoint placement of graphs*, Journal of Combinatorial Theory, Series B **25** (1978), no. 3, 295–302.

[13] B. Sudakov and J. Volec, *Properly colored and rainbow copies of graphs with few cherries*, Journal of Combinatorial Theory, Series B **122** (2017), 391–416.